§ Riemann Curvature Tensor

Motivation: $S^2 \subseteq \mathbb{R}^3 \rightsquigarrow$ Gauss curvature K "intrinsic" & Mean curvature H "extrinsic"

Q: What is the "appropriate" notion of curvature for (M, g)?

Note: "higher dim" & "intrinsic".

A: Riem. curvature tensor. $\text{Riem.} = R$

Def$^{\underline{n}}$: The Riemann curvature of (M^n, g) is an association to each $X, Y \in \mathcal{T}(TM)$ a map

$$R(X, Y) : \mathcal{T}(TM) \longrightarrow \mathcal{T}(TM)$$

defined by

$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

Levi-Civita connection

Remark: $R(X,Y)Z$ are linear in X, Y and Z.

Prop: $R(X,Y)Z$ are "tensorial" in X, Y and Z.

i.e. $R(fX, Y)Z = R(X, fY)Z = R(X,Y)(fZ) = f\,R(X,Y)Z$ $\quad \forall f \in C^\infty(M)$

Proof: Note: $R(X,Y) = -R(Y,X)$, since $[Y,X] = -[X,Y]$.

So it suffice to show $R(fX, Y)Z = f(R(X,Y)Z)$,

and $R(X,Y)(fZ) = f(R(X,Y)Z)$.

$\underline{R(fX,Y)Z = f\,R(X,Y)Z:}$

$$R(fX,Y)Z = \nabla_Y \nabla_{fX} Z - \nabla_{fX}\nabla_Y Z + \nabla_{[fX,Y]} Z.$$

$$= \nabla_Y (f\nabla_X Z) - f\,\nabla_X \nabla_Y Z + \nabla_{f[X,Y] - Y(f)X} Z$$

$$= f\,\underline{\nabla_Y \nabla_X Z} + \cancel{Y(f)\nabla_X Z} - f\,\underline{\nabla_X \nabla_Y Z}$$

$$+ f\,\underline{\nabla_{[X,Y]} Z} - \cancel{Y(f)\,\nabla_X Z}.$$

$$= f\,R(X,Y)Z$$

$\underline{R(X,Y)(fZ) = f\,R(X,Y)Z}$:

$$R(X,Y)(fZ) = \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) + \nabla_{[X,Y]}(fZ)$$

$$\nabla_Y \nabla_X (fZ) = \nabla_Y (f\nabla_X Z + X(f)Z)$$

$[Y,X](f)Z$

$\parallel$

so $\nabla_Y \nabla_X (fZ) = f\nabla_Y \nabla_X Z - \cancel{Y(f)\nabla_X Z} + \cancel{X(f)\,\nabla_Y Z} + \boxed{YX(f)\,Z.}$

$$- \quad \nabla_X \nabla_Y (fZ) = f\nabla_X \nabla_Y Z - \cancel{X(f)\nabla_Y Z} + \cancel{Y(f)\,\nabla_X Z} + \boxed{XY(f)\,Z.}$$

$$+ \quad \nabla_{[X,Y]}(fZ) = \boxed{\cancel{[X,Y](f)\,Z}} + f\,\nabla_{[X,Y]} Z.$$

$$\text{R.H.S.} = f\,R(X,Y)Z$$

$\blacksquare$

Def^n: $R(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle$

(0,4) - tensor
Riem. curvature tensor

Prop: (Symmetries of Riem. curvature tensor)

(a) Bianchi identity :

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$$

cylic permutation

(b) $$R(X,Y,Z,W) = -R(Y,X,Z,W)$$

(c) $$R(X,Y,Z,W) = -R(X,Y,W,Z)$$

(d) $$R(X,Y,Z,W) = R(Z,W,X,Y)$$

Proof: (a)

$$R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$$

$$+\; R(Y,Z)X = \nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X + \nabla_{[Y,Z]}X$$

$$+\; R(Z,X)Y = \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y + \nabla_{[Z,X]}Y$$

$$\text{R.H.S.} = \nabla_Y[X,Z] + \nabla_Z[Y,X] + \nabla_X[Z,Y] - \nabla_{[X,Z]}Y - \nabla_{[Y,X]}Z - \nabla_{[Z,Y]}X$$

$$= [Y,[X,Z]] + [Z,[Y,X]] + [X,[Z,Y]] = 0$$

"Jacobi identity"

(b) $R(X,Y)Z = -R(Y,X)Z$. done!

(c) It suffices to show $R(X,Y,T,T) = 0$ ($\because$ set $T = W + Z$)

$$R(X,Y,T,T) = \langle R(X,Y)T, T\rangle$$

$$= \langle \nabla_Y\nabla_X T - \nabla_X\nabla_Y T + \nabla_{[X,Y]}T ,\ T\rangle$$

$$= \langle \nabla_Y\nabla_X T, T\rangle - \langle \nabla_X\nabla_Y T, T\rangle + \langle \nabla_{[X,Y]}T, T\rangle$$

Note: $\langle \nabla_Y \nabla_X T, T\rangle = Y\langle \nabla_X T, T\rangle - \cancel{\langle \nabla_X T, \nabla_Y T\rangle}$

$- \quad \langle \nabla_X \nabla_Y T, T\rangle = X\langle \nabla_Y T, T\rangle - \cancel{\langle \nabla_Y T, \nabla_X T\rangle}$

$+ \quad \langle \nabla_{[X,Y]} T, T\rangle = \frac{1}{2}[X,Y]\langle T, T\rangle$

$$\text{R.H.S.} = Y\left(\tfrac{1}{2}X\langle T,T\rangle\right) - X\left(\tfrac{1}{2}Y\langle T,T\rangle\right) + \tfrac{1}{2}[X,Y]\langle T,T\rangle$$

$$= 0$$

(d) Bianchi $\Rightarrow$

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$$

$$+ \quad R(Y,Z,W,X) + R(Z,W,Y,X) + R(W,Y,Z,X) = 0$$

$$+ \quad R(Z,W,X,Y) + R(W,X,Z,Y) + R(X,Z,W,Y) = 0$$

$$+ \quad R(W,X,Y,Z) + R(X,Y,W,Z) + R(Y,W,X,Z) = 0$$

Cancels

$$2\left(R(Z,X,Y,W) + R(W,Y,Z,X)\right) = 0$$

$$\Rightarrow R(Z,X,Y,W) = R(Y,W,Z,X)$$

$\blacksquare$

In local coord. of M, say $(x^1, \dots, x^n)$, let $\partial_i := \frac{\partial}{\partial x^i}$

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle = \langle \partial_i, \partial_j\rangle$$

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

Compute $R(\partial_i, \partial_j, \partial_k, \partial_l) =: R_{ijkl}$

$$\nabla_{\partial_j} \nabla_{\partial_i} \partial_k = \nabla_{\partial_j}(\Gamma_{ik}^{l} \partial_l) = (\partial_j \Gamma_{ik}^{s}) \partial_s + \Gamma_{ik}^{l} \Gamma_{jl}^{s} \partial_s$$

i.e. $\nabla_{\partial_j} \nabla_{\partial_i} \partial_k = [\partial_j \Gamma_{ik}^{s} + \Gamma_{ik}^{l} \Gamma_{jl}^{s}] \partial_s$

Similarly, $\nabla_{\partial_i} \nabla_{\partial_j} \partial_k = [\partial_i \Gamma_{jk}^{s} + \Gamma_{jk}^{l} \Gamma_{il}^{s}] \partial_s$

and $\nabla_{[\partial_i, \partial_j]} \partial_k = 0$

$$\Rightarrow R(\partial_i, \partial_j, \partial_k, \partial_l) = g_{sl}\left(\partial_j \Gamma_{ik}^{s} - \partial_i \Gamma_{jk}^{s} + \Gamma_{ik}^{l} \Gamma_{jl}^{s} + \Gamma_{jk}^{l} \Gamma_{il}^{s} \right)$$

i.e. $R_{ijkl} = g_{sl}(\partial \Gamma + \Gamma * \Gamma) = F(g, \partial g, \partial^2 g)$

<u>Symmetries</u>
$$\begin{cases} R_{ijkl} + R_{jkil} + R_{kijl} = 0 & \text{(Bianchi)} \\ R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij} \end{cases}$$

Q: How is the Riemann curvature tensor R related to the notion of "Gauss curvature" for surfaces in $\mathbb{R}^3$?

A: "sectional curvature"

Fix $p \in M$, and a 2-dim'l subspace $\sigma \subseteq T_pM$

Def^{n}: Sectional curvature of σ at $p \in M$ is defined as

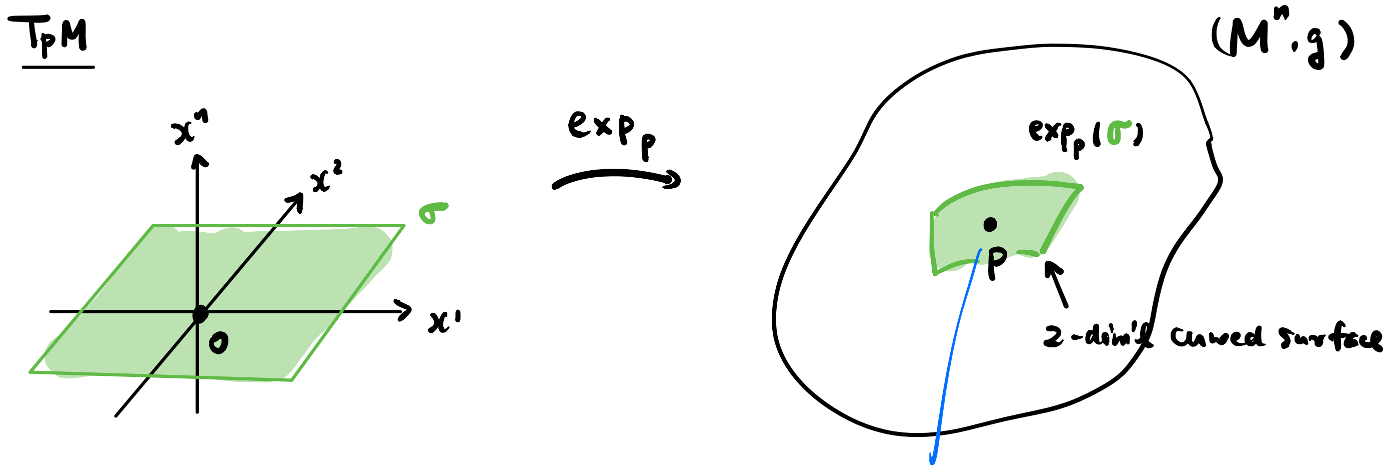
$$K_p(\sigma) := R(e_1, e_2, e_1, e_2)$$

where $\{e_1, e_2\}$ O.N.B. for σ.

FACT: $K(\sigma)$ is "well-defined", ie indep. of the choice of O.N.B $\{e_1, e_2\}$.

Geometric Meaning: $K_p(\sigma) \in \mathbb{R}$ measures the Gauss curvature at p of a "sub-surface" generated by σ in M.

geodesic normal coord.



(Proof: Exercise!)

Gauss curvature of this sub-surface at $p = K_p(\sigma)$

We have the following algebraic fact.

Prop: Knowing all the sectional curvatures $K_p(\sigma)$ for all $\sigma \subseteq T_pM$ determines completely the Riem. curvature tensor R at p.

Proof: Idea: $R_{ijij} \overset{\text{determines}}{\rightsquigarrow} R_{ijk\ell}$ using symmetries of R

Let $\{e_1, \dots, e_n\}$ be an O.N.B. for T_pM,

$$\sigma_{ij} := \operatorname{span}\{e_i, e_j\} \subseteq T_pM \quad , \quad i \neq j.$$

$$K(\sigma_{ij}) := R(e_i, e_j, e_i, e_j).$$

Using multi-linearity, only need to know $R(e_i, e_j, e_k, e_l)$, $\begin{smallmatrix} i \neq j \\ k \neq l \end{smallmatrix}$.

Note: $R\left(\frac{e_i+e_k}{\sqrt{2}}, e_j, \frac{e_i+e_k}{\sqrt{2}}, e_j\right) = K\left(\operatorname{span}\left\{\frac{e_i+e_k}{\sqrt{2}}, e_j\right\}\right)$

BUT $R(e_i+e_k, e_j, e_i+e_k, e_j)$

$$= \underbrace{R(e_i, e_j, e_i, e_j)}_{K(\sigma_{ij})} + \underbrace{R(e_k, e_j, e_k, e_j)}_{K(\sigma_{kj})}$$

$$+ R(e_i, e_j, e_k, e_j) + R(e_k, e_j, e_i, e_j)$$

same

$\Rightarrow \quad R(e_i, e_j, e_k, e_j) =$ "known"

Note: $R\left(e_i, \frac{e_j+e_l}{\sqrt{2}}, e_k, \frac{e_j+e_l}{\sqrt{2}}\right) =$ "known"

BUT $R(e_i, e_j+e_l, e_k, e_j+e_l)$

$$= \underbrace{R(e_i, e_j, e_k, e_j)}_{\text{"known"}} + \underbrace{R(e_i, e_l, e_k, e_l)}_{\text{"known"}} \quad - R(e_j, e_k, e_i, e_l)$$

$$+ R(e_i, e_j, e_k, e_l) + \underbrace{R(e_i, e_l, e_k, e_j)}$$

i.e. $\quad R(e_i, e_j, e_k, e_l) - R(e_j, e_k, e_i, e_l) =$ "known"

$-) \quad R(e_k, e_i, e_j, e_l) - R(e_i, e_j, e_k, e_l) =$ "known"

$2R(e_i, e_j, e_k, e_l) + R(e_i, e_j, e_k, e_l) =$ "known"

$\blacksquare$

Cor: Let $c \in \mathbb{R}$ be a constant.

$K(\sigma) \equiv c$ $\forall \sigma \subseteq T_pM$ $\iff$ $R(X, Y, Z, W) = c\,(\langle X, Z\rangle\langle Y, W\rangle - \langle Y, Z\rangle\langle X, W\rangle)$

§ Ricci and scalar curvature

Let $\{e_1, \dots, e_n\}$ be an O.N.B. for T_pM.

<u>Def$^{\underline{n}}$</u>: Ricci curvature $\quad \mathrm{Ric}(X,Y) := \sum_{i=1}^n R(X, e_i, Y, e_i)$

Scalar curvature $\quad S := \sum_{i=1}^n \mathrm{Ric}(e_i, e_i)$

<u>FACT</u>: well-defined, indep. of choice of O.N.B. $\{e_1, \dots, e_n\}$

In local coord.,

$$R_{ijkl} \xrightarrow{\text{"trace"}} R_{ik} := g^{jl} R_{ijkl} \xrightarrow{\text{"trace"}} R := g^{ik} R_{ik}$$

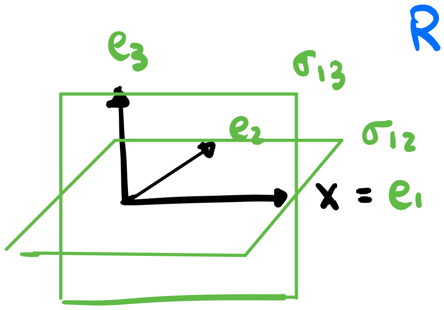
<u>Riem</u> (0,4)-tensor

<u>Ricci</u> (0,2)-tensor "symmetric"

<u>Scalar</u> function

<u>Geometric meaning</u>: Ric & S are "averaged" sectional curvatures:

O.N.B. $\{X = e_1, e_2, \dots, e_n\}$



$$\mathrm{Ric}(X,X) = \mathrm{Ric}(e_1,e_1) := \sum_{i=1}^n R(e_1, e_i, e_1, e_i)$$

$$= \underbrace{\sum_{i=2}^n \overbrace{R(e_1, e_i, e_1, e_i)}^{K(\sigma_{1i})}}_{\substack{\text{sum of sect. curv.} \\ \text{of planes through } e_1 = X}}$$

Similarly, $\quad S := \sum_{i=1}^n \mathrm{Ric}(e_i, e_i) = \sum_{i=1}^n \left(\sum_{j=1}^n R(e_i, e_j, e_i, e_j) \right)$

$$= \underbrace{\sum_{i \neq j} R(e_i, e_j, e_i, e_j)}_{\text{sum of all sectional curv.}}$$

A Central Question in Riemannian Geometry

How does the Riem / Ric / Scalar curvatures affect the local / global geometry of (M^n, g) ?

E.g.) Gauss-Bonnet Thm: $\iint_S K\, da = 2\pi \chi(S)$.

Now, we digress a bit to talk about covariant derivatives of general tensors ......

Recall: A connection ∇ induces a covariant derivative for vector fields (i.e. $(1,0)$-tensor):

Fix $X \in \Gamma(TM)$.

$$\begin{aligned} \nabla_X : \Gamma(TM) &\longrightarrow \Gamma(TM) \\ Y &\longmapsto \nabla_X Y \end{aligned}$$

Q: How to covariant differentiate other tensors?

A: "Liebniz rule"

(i.e. $(0,1)$-tensors)

1-forms : $\omega \in \Omega^1(M) = \Gamma(T^*M) \rightsquigarrow \nabla_X \omega \in \Omega^1(M)$ defined as

$$(\nabla_X \omega)(Y) := X(\underbrace{\omega(Y)}_{\text{function}}) - \omega(\underbrace{\nabla_X Y}_{\text{v.f.}})$$

v.f. 1-form v.f. v.f. 1-form

(1,1)-tensors : $\alpha \in \Gamma(T^1_1 M) \rightsquigarrow \nabla_X \alpha \in \Gamma(T^1_1 M)$ defined as

$$(\nabla_X \alpha)(Y, \omega) := X(\alpha(Y, \omega)) - \alpha(\nabla_X Y, \omega) - \alpha(Y, \nabla_X \omega)$$

Example 1: (M^n, g) $\quad g$: $(0,2)$-tensor $\rightsquigarrow$ $\exists!$ connection ∇

metric compatibity $\Longleftrightarrow$ $\boxed{\nabla g \equiv 0}$ ie $\nabla_X g \equiv 0 \quad \forall X$

Why?

$$\underbrace{(\nabla_X g)(Y,Z)}_{\nabla g \equiv 0 \quad \overset{\parallel}{0}} := \underbrace{X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)}_{\overset{\parallel}{0} \quad \text{metric compatibility}}$$

Example 2: (Riem. curvature acting on 1-form)

Let $\omega \in \Omega^1(M)$. Define:

$$R(X,Y)\,\omega := \nabla_Y \nabla_X \omega - \nabla_X \nabla_Y \omega + \nabla_{[X,Y]}\omega$$

FACT: $\left(R(X,Y)\,\omega\right)(Z) = -\omega\left(R(X,Y)Z\right)$

Pf: $\left(R(X,Y)\,\omega\right)(Z)$

$$= \left(\nabla_Y \nabla_X \omega - \nabla_X \nabla_Y \omega + \nabla_{[X,Y]}\omega\right)(Z)$$

$$= Y\left((\nabla_X \omega)(Z)\right) - (\nabla_X \omega)(\nabla_Y Z)$$
$$- X\left((\nabla_Y \omega)(Z)\right) + (\nabla_Y \omega)(\nabla_X Z)$$
$$+ [X,Y]\left(\omega(Z)\right) - \omega\left(\nabla_{[X,Y]} Z\right)$$

$$= Y\left(X(\omega(Z)) - \omega(\nabla_X Z)\right) - X\left(\omega(\nabla_Y Z)\right) + \omega(\nabla_X \nabla_Y Z)$$
$$- X\left(Y(\omega(Z)) - \omega(\nabla_Y Z)\right) + Y\left(\omega(\nabla_X Z)\right) - \omega(\nabla_Y \nabla_X Z)$$
$$+ [X,Y]\left(\omega(Z)\right) - \omega\left(\nabla_{[X,Y]} Z\right)$$

$$= -\omega\left(R(X,Y),Z\right).$$

$\blacksquare$

1^{st} Bianchi identity : $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$

2^{nd} Bianchi identity : $(\nabla_X R)(Y,Z,W,T) + (\nabla_Y R)(Z,X,W,T)$

$$+ (\nabla_Z R)(X,Y,W,T) = 0$$

(Pf: Exercise!)